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ON DESCRIPTION OF DYNAMIC PROPERTIES OF THE GRAVITATIONAL FIELD IN VACUUM*

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In constructing dynamic physical models of the gravitational field within the limits of continua defined by the four-dimensional Riemennian space-time with local structure of pseudo-Euclidean Minkovski space, we first consider the methods of introducing certain kinematic characteristics of such spaces.

Four sets of coordinate lines correspond to any global coordinate system. It is generally possible to isolate coordinate systems, in which one of the coordinate sets consists of time-like or isotropic lines. Each line of such set may be taken as the world line of individual points that form a three-dimensional manifold of points of an "ideal medium".

Various coordinate systems can be taken as a reference system for the system of world lines that define the ideal medium. The simplest example of such reference system is that of world lines of moving points of various material continua for which the respective coordinate systems are represented by accompanying systems of Lagrangian coordinates. Various coordinate systems with different remaining three sets of coordinate lines can be introduced for the same system of world lines taken as the reference system; it is, also, possible to introduce the respective reference system for every system of timelike or isotropic coordinate lines.

A unit vector tangent to the timelike world line represents by definition the four-dimensional velocity of ideal medium points; it is also possible to introduce for isotropic elements of world lines four-dimensional tangent vectors whose length is, however, zero.

It is possible to use in any Riemannian space a large variety of ideal media and corresponding reference systems, such as reference systems of particular forms to suit various kinds of Riemannian surface symmetry, reference systems with geodetic world lines, in particular, synchronous coordinate systems, systems with isotropic world lines including those with geodetic isotropic lines, systems of harmonic coordinates, etc. In each of these examples the respective reference systems are, generally, not uniquely related to the Riemannian space.

A special accompanying reference system for the ideal medium, or the corresponding system of tetrads uniquely and invariantly related to the Riemannian space is effectively introduced below.

Two accompanying reference systems can be introduced: one for the observer, and the other in the form of a Lagrangian system for some mobile model system.

The characteristics and events determined and defined by the Lagrangian system for the model medium can, also, be analyzed using a coordinate system attached to the observer's coordinate system. The conversion of results obtained theoretically or experimentally in the accompanying Lagrangian system to those theoretically or experimentally obtained in that of the observer's represents a navigational problem generally involving integration of systems of equations in partial derivatives, which are closely related to the use of the determination of the reference and of the observer's coordinate systems /1,2/.

For every reference system it is possible to consider a four-dimensional infinitely thin filament formed by infinitely close world lines adjacent to some selected fixed world line C.

Let us consider an infinitely small volume dV that represent the locus of the ends of spatially similar vectors $d\mathbf{l}$ drawn from points of line C at a given instant of the proper time τ on C and orthogonal to C. Let at the variation $\tau'(\tau' > \tau)$ the end points of vector $d\mathbf{l}$ move along their adjacent world lines C'. At the instant of proper time $\tau' > \tau$ the volume dV is replaced by volume dV', and the respective points dV and dV' are particularized by their world lines, in other words, by the same values of Lagrangian coordinates. As the result of this transformation, the infinitely small vector $d\mathbf{l}$ becomes vector $d\mathbf{l}'$. The transformation of the infinitely small three-dimensional space volume dV to dV' is a point affine transformation that can be defined as a pure finite deformation with a finite angle of rotation of the principal deformation axes. When $\tau' - \tau = d\tau$ is infinitely small it is possible to introduce the respective deformation rate tensors and the components of the corresponding rotation rate asymmetric tensor. It is moreover possible to introduce and calculate the components of the Riemannian space curvature tensor using these characteristics, and generally to obtain expressions for the four-dimensional accelerations of points on world lines and

*Prikl. Matem.Mekhan.,44,No.2,195-204,1980 formulas for the deviation of adjacent world lines. The indicated characteristics are of a kinematic character whose specific form and properties are determined by the properties of the reference system which in any given finite Riemannian space region can be introduced with considerable arbitrariness. As this arbitrariness is reduced, the reference system characteristics become dependent on the geometric properties of the Riemannian space. The transformation of dV to dV' and the respective characteristics of space and of the reference system world lines are, thus, closely related to the selection of a reference system, i.e. of the ideal medium model, which, in turn, is related to the Riemannian space and could be considered as the ideal medium that accompanies the Riemannian space.

The construction of such accompanying reference system is equivalent to the construction of the related velocity field with the elimination of possible arbitrariness, otherwise, to the construction of a vector field uniquely and invariantly linked with the geometric characteristics that determine the Riemannian space properties.

The construction of such a model and of the accompanying reference system can be based on Petrov's theory /3,4/, who had investigated the problem of the Weyl tensor canonical form defined by a simple formula in terms of the Ricci, the metric, and the Riemann tensors. If the Ricci tensor is zero, the Weyl tensor is the same as the Riemann tensor, and, consequently, it completely determines in Einstein's dynamical theory the essential properties of the gravitational field in vacuum. In the finite region in which the Ricci and Weyl tensors are zero, the space is pseudo-Euclidean.

According to Petrov's theory it is generally possible to indicate at every point of the Riemannian space tetrads with uniquely determined four mutually perpendicular directions in which the independent Weyl tensor components reduce to the canonical form, and in the general case represent according to him four independent real invariants. The respective canonical forms of the Weyl tensor determine at a given point a Riemannian space of the algebraic type.

If a corresponding six-dimensional bivector space is introduced for the Weyl tensor, the first type corresponds to the case when all roots of the respective secular equations are different, if these roots are multiple, special degenerate types of the Weyl tensor obtain with corresponding degenerate Riemannian spaces of type 2, 3, N, and D.

According to the theory developed by Debever, Sachs, Pirani, Newman, and Penrose /5—9/ it is possible to introduce in a unique manner at points of type 1 space, four different principal isotropic directions, of which two merge at points of type 2 space, at points type 3 space the three principal isotropic directions merge, and in type N all four principal isotropic directions merge. Type D obtains when the four principal isotropic directions merge pairwise in two isotropic directions.

It was proposed in /2/ to take as the vector determining the accompanying reference system in type 1 the uniquely determined timelike basis vector э_4 directed into the future in tetrads that, according to Petrov, are determined by the canonical directions for the Weyl tensor. Zhukov had shown /10/ that in the continuous transition from the first type to special degenerate types the direction of vector э_4 continuously passes to the merged principal isotropic directions that are uniquely determined, and that only in type D two such isotropic directions obtain in a four-dimensional space.

An accompanying reference system has been thus derived for all types of Riemannian space when the Weyl tensor is nonzero. Properties of respective world lines and their characteristics indicated above are essential invariant properties of the space itself, and in the case of the gravitational field in vacuum (when the Ricci tensor is zero) the system of canonical tetrads and the Weyl tensor components invariant in them completely determine the geometry of the four-dimensional spacetime. This clearly shows that the dynamic (physical) properties of the gravitational field are generally determined by the properties of the devised system of canonical tetrads and the four independent invariants, for instance, the four independent Weyl tensor components in these tetrads.

It is, consequently, reasonable and evidently convenient or necessary to devise physical models of the gravitational field in vacuum using such a reference system. The respective expressions for energy, the energy-momentum tensor, and other scalar and tensor characteristics of the gravitational field in the devised reference system attached to the gravitational field would be, generally speaking, of a simpler form than in other reference systems, since it is in this system that the various extraneous properties and characteristic parameters, which have to be necessarily introduced in any other reference system, are eliminated.

It is remarkable that the reference system devised here attached to the field in vacuum in degenerate types in which the respective world lines become isotropic geodesic lines that correspond to merged principal isotropic directions, had been earlier introduced and applied by many authors for qualitative investigations of the gravitational field properties, and for devising a method for deriving exact solutions of dynamic equations of the gravitational field /11—16/. It was shown in /9/ that in such reference systems composed of isotropic geodesic world lines the field equations are of the simplest form which permits direct integration and makes it possible to obtain the majority of known exact solutions, as well as a number of new exact solutions. These methods were, however, used up to the present only for defining spaces of the degenerate type.

Reference systems constructed of arbitrary isotropic or nonisotropic goedesic world lines are in the general case, as well as in that of degenerate types, not uniquely related to the invariant geometric singularities of space.

In canonical tetrads, or in the accompanying global coordinate system constructed with their use, the Weyl tensor components are of special form. This property must be considered as the condition that defines canonical tetrads and the selected reference system attached to the gravitational field, as well as the ideal medium for simulating the gravitational field. Owing to the special properties of the Weyl tensor components, the kinematic relations for respective world lines are also of special form that corresponds to essential singularities of the gravitational field.

As an example, let us consider the classical problem of the gravitational field energy and of its energy-momentum tensor in the region of vacuum, i.e. in that part of the space where matter and the elctromagnetic field are absent.

The definition of the gravitational field in vacuum makes it possible to consider it as a Riemannian space with zero Ricci tensor. This corresponds to the vacuum dynamic equations (without the cosmological term) of Einstein's theory.

The dynamic equations in various variants of the theory of gravitation are nonlinear equations in partial derivatives. For solving these in the case of a finite volume we have no known physically proved boundary conditions and no convincingly substantiated conditions at possible strong surface discontinuities.

The formulation of conditions at the boundaries of the continuity region is closely related to conditions at strong discontinuities whose establishment is always associated with the stipulation of additional assumptions which, in turn, are associated with the determination of equations of state and the possible presence of some additional physical processes and effects at discontinuities. Such additional effects and the respective equations of state, when used in connection with variational principles, may differ from the specified invariably fixed Euler equations which form a closed system. Particular solutions, solutions with specified or unknown asymptotics at singular points that can be considered as boundary elements of the region of continuous solutions may be sought for the system of Euler equations.

The condition that in the case of vacuum the Ricci tensor must be zero completes the Euler equations; further derivation of specific solutions depends on and is predetermined explicitly or implicitly by the boundary conditions.

The establishment of conditions at strong discontinuities is, thus, a supplementary physical problem of a nature similar to that of the basic problem of formulating Euler's equations for continuous processes and motions in finite regions occupied by material media and an electromagnetic field. Methods of constructing models of material media and fields using the basic variational equation, which is a generalization of the local energy equation and of universal and special thermodynamic relationships for the considered specific media, were developed in a number of publications /17—22/. These methods make it possible to establish not only the Euler equations but, also, the equations of state together with conditions at strong discontinuities.

The application of the methods of model construction developed in this way makes possible a clear understanding and definition of the situation arising in connection with the assumptions proposed by various authors relative to the gravitational field pseudotensors which from the physical point of view are unsatisfactorily determined. It also offers a new approach to a physically acceptable solution of the problem of determination of the gravitational field energy as a scalar and the description of interactions in that field that are determined by real tensors.

With the availability of a system of accompanying tetrads for the gravitational field with a corresponding reference system attached to the ideally determined medium, it is possible to introduce by invariant means and without affecting the field system of equations, the concept of specific energy as a four-dimensional scalar by adding to the Lagrangian Λ in the previously introduced basic variational equation the divergence of some four-dimensional vector Ω.

We introduce tetrads with basic vectors $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4$, where vector $\mathfrak{s}_4$ is tangent to the world lines of the attached reference system defined above; that vector may be assumed to be a unit vector in type 1 directed along the merged principal isotropic directions in the deggenerate types. In type 1 vector $\mathfrak{s}_4$ is uniquely determined, while vectors $\mathfrak{s}_1, \mathfrak{s}_2$ and $\mathfrak{s}_3$ oriented along canonical directions are, according to Petrov, accurate only as regards their sign. In degenerate types vector $\mathfrak{s}_4$ is isotropic and vector $\mathfrak{s}_3$ may be assumed isotropic. Vectors $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are orthogonal to $\mathfrak{s}_3$ and $\mathfrak{s}_4$. Hence the following normalization and orthogonality conditions apply:

$$
\begin{gathered}
(\mathbf{э}_4, \mathbf{э}_4) = (\mathbf{э}_3, \mathbf{э}_3) = 0, (\mathbf{э}_3, \mathbf{э}_4) = 1, \ (\mathbf{э}_1, \mathbf{э}_1) = (\mathbf{э}_2, \mathbf{э}_2) = 1 \\
(\mathbf{э}_4, \mathbf{э}_1) = (\mathbf{э}_4, \mathbf{э}_2) = (\mathbf{э}_3, \mathbf{э}_1) = (\mathbf{э}_3, \mathbf{э}_2) = (\mathbf{э}_2, \mathbf{э}_1) = 0
\end{gathered} \tag{1}
$$

It was shown by A. V. Zhukov that in the transition form one type to another, the direction of vector $\mathbf{э}_4$ is continuous, while the continuity conditions cannot be satisfied for the directions of vectors $\mathbf{э}_1, \mathbf{э}_2, \mathbf{э}_3$. It should be pointed out that under condition (1) the directions of vectors $\mathbf{э}_1, \mathbf{э}_2, \mathbf{э}_3$ are not uniquely determined in degenerate types. The selection of vector $\mathbf{\Omega}$ which by definition can be introduced in the indicated tetrads represents an additional essential postulate which, owing to its nature, can be put in the same category as the known fundamental postulates of the general theory of relativity. This postulate, as other postulates applied in the simulation of real gravitational phenomena in nature, are in need of experimental verification. This postulate does not in essence affect the basic equations of the field theory, but makes itself felt through boundary conditions and conditions at strong discontinuities that may accompany the considered phenomena. (In actual derivation of solutions of dynamic equations the supplementary conditions that replace boundary conditions are always postulated, either explicitly or implicitly).

The assumption that strong discontinuities can always be removed, albeit at the cost by introducing complications of physical properties of basic models and correlate Eulers' equations systems, (the complications appear only in the narrow regions of strong discontinuities in the simplified models). This does not eliminate the problem of spreading of the volume gravitational energy intrinsic to the field in intermediate regions, where simple models are acceptable, and also the problem of emission and absorption of energy because of singularities in the field or at its boundaries.

There are no reasons at present for complicating many of the existing models of media and, consequently, it is possible to begin the investigation of energy density ε of the gravitational field in vacuum on the basis of the described properties of canonical tetrads and requirements for physical acceptability of obtained results, assuming that

$$
\varepsilon = \nabla_i \Omega^i \tag{2}
$$

and consider functions Ω^i to be components of vector $\mathbf{\Omega}$. In the simplest field model we can assume that

$$
\mathbf{\Omega} = k\mathbf{э}_4 \tag{3}
$$

where k can be a scalar function of invariants of the Weyl tensor coincident with the Riemann tensor or, even simpler, to assume k to be some physical constant of dimension $L\cdot[\Lambda]$, where L is length. Assumption (2) is usually accepted by all authors who postulate, often implicitly, the expression for the gravitational field energy, but the introduced functions Ω^i are not taken as components of some vector. Once formulas (2) and (3) have been fixed, further problems related to the tensor of the gravitational field energy-momentum and to conditions at strong discontinuities are solved using the technique developed for these problems /21/. If $k = \text{const}$ is assumed in formula (3), then in any tetrad locally determined as an inertial reference system with orthogonal axes, in which the basis vector is equal to vector $\mathbf{э}_4$ defined above, the equality $\mathbf{\Omega} = k\mathbf{э}_4$ is satisfied and vector $\mathbf{\Omega}$ is directed along the tangent to world lines in the accompanying reference system devised above. The accompanying reference system corresponds to the system of inertial tetrads determined at every point of space. These tetrads are determined at every point of space with an accuracy to spatial rotation and reflection of space axes. They can have principal axes conforming to Petrov, or Fermi—Walker tetrads /23/, or some other tetrad system depending on the law of rotation of the spatial tetrad axes during transition from one point to another along each world line.

According to (3), vector $\mathbf{\Omega}$ is constant in each of such inertial tetrad systems, since it can vary only in consequence of a change of the tetrad timelike axis $\mathbf{э}_4$. We denote by $\mathbf{э}_i$ the orthonormalized basis vectors at point M, and, taking into consideration the inertial properties of local reference systems in tetrads, denote these vectors at point M' by $\mathbf{э}_i'$; these vectors can be transferred from point M' to point M. The orthonormalized bases $\mathbf{э}_i$ and $\mathbf{э}_i'$ are evidently linked at point M by the infinitely small Lorentz transformation of the form $\mathbf{э}_i' = (\delta_{\cdot i}^{j} + \gamma_{\cdot il}^{j\cdot\cdot} y^l)\mathbf{э}_j$, where y^l are the infinitely small coordinates of point M' in the tetrad corresponding to point M and $\gamma_{jil} y^l$ is an antisymmetric matrix with respect to i and j: $\gamma_{ijl} = -\gamma_{jil}$. Twenty four independent components of tensor γ_{ijl} are called Ricci symbols and determine a system of tetrads. Components of the curvature tensor and all characteristics in the accompanying reference system can be expressed in terms of γ_{ijl}

Equality (3) implies that the formulas

$$
\mathbf{\Omega} = \Omega^i \mathbf{э}_i' = k\,(\delta_{4\cdot}^{\cdot i} + \gamma_{\cdot 4l}^{i\cdot\cdot} y^l)\,\mathbf{э}_i; \quad \nabla_i \Omega^i = k\gamma_{\cdot 4i}^{i\cdot\cdot} \tag{4}
$$

are valid in the tetrad corresponding to point M.

The addition of divergence $\nabla_i \Omega^i$ to the basic variational equation results in the appearance in the latter of the following two balanced terms:

$$
\delta \int_{V_4} \nabla_i \Omega^i \, d\tau + \delta W_\Omega = 0
$$

If there is no discontinuity inside V_4, then, according to formula (10) in /2/ it is possible to write

$$\delta W_\Omega = -\int_{\Sigma_3}\left[\nabla_k\Omega^k\delta x^i + \frac{1}{\sqrt{-g}}\partial(\sqrt{-g}\Omega^i)\right]n_i\,d\sigma \tag{5}$$

By applying locally in the tetrad reference system formulas (4) and (5) to the small volume dV_4 which does not contain discontinuities and is bounded by the closed surface $d\Sigma_3$, and taking into account the equality $\sqrt{-g} = -1$ valid in the tetrad coordinate system, we obtain

$$\delta\int_{dV_4}\nabla_i\Omega^i\,d\tau = -\,\delta W_\Omega = \int_{d\Sigma_3}(\nabla_k\Omega^k\delta x^i + \partial\Omega^k\delta^{i}_{.k})\,n_i\,d\sigma = -\int_{d\Sigma_3}S_{j.}^{\cdot i}\delta y^j n_i\,d\sigma \tag{6}$$

From which we have

$$S_{j.}^{\cdot i} = k\,(\gamma^{i\cdot\cdot}_{\cdot qj} - \gamma^{q\cdot\cdot}_{\cdot qq}\delta^{i}_{\cdot j}) \tag{7}$$

The components of tensor $S_{j.}^{\cdot i}$ may be considered as the components of the energy-momentum tensor of the gravitational field in vacuum.

If inside dV_4 there is a strong discontinuity surface S_k, then the term $[S_{j.}^{\cdot i}\delta y^j n_i d\sigma_k]_{S_k}{\pm}$ appears in the right-hand side of equality (6). This additional term together with other similar terms dependent on the material medium and the electromagnetic field defined in the tetrad coordinate system must be taken into account in conditions at discontinuities that follow from the basic variational equation.

When the continuity condition of δy^j is satisfied, we have for discontinuities in the gravitational field in vacuum the following conditions on the surface S_k:

$$[S_{j.}^{\cdot i} n_i\,d\sigma_k] = 0 \tag{8}$$

The presence of the additional terms $\nabla_i\Omega^i$ in the expression for the Lagrangian and components of tensor $S_{j.}^{\cdot i}$ may appear in equations of state, when solving specific problems, only in terms of boundary or initial conditions and conditions at discontinuities.

Formulas (6) and (7) are expressed in tetrad bases, while formula (8) holds in any coordinate system. The expression for components $S_{j.}^{\cdot i}$ can be transformed for any coordinate system using the rules of tensor transformations and the formulas for passing from the specified coordinate system to the locally selected tetrad system obtained from the respective definitions of these coordinate systems.

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